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On the energy shape dependences of ellipsoidal leptodermous systems

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Abstract. We study the shape-dependent functions that determine the energy of charge (or mass) ellipsoidal distributions. Investigating the symmetries of the ellipsoid—invariance against axis relabelling and transformation to its reciprocal—we show (i) that the mean curvature of an ellipsoid is strongly connected to the area of its reciprocal and (ii) that the energy can be deduced analytically from only one angular moment.

1. Introduction

Among the equilibrium figures that can be attained by a distribution of matter or charge held together by a surface tension, the ellipsoidal shapes occur frequently.

As early as 1834 Jacobi concluded 'that ellipsoids with three unequal axes can very well be figures of equilibrium'. Such ellipsoidal figures of equilibrium may be induced, at least approximately, either by rotation (planets, ellipsoidal galaxies, classical rotating liquid drops and nuclei with high angular momenta) or by the self-consistent field of an N-body system (nuclei with low spins). The ellipsoidal shapes—viewed as small deformations from the spherical ones—are interesting, because one can use the considerable historical studies on the ellipsoid properties, a review of which can be found in Chandrasekhar (1969). Furthermore, in nuclear physics, the dynamical studies of the nuclear deformation require that we do not restrict ourselves to axial symmetries (Kumar 1975).

Many physical distributions of matter or charge can be considered as leptodermous (thin skin) systems; in that case, the localisation of the major modifications of the energy, caused by the surface, on a relatively thin surface region, allows analytical developments of the energy expression. The most sophisticated analysis of this kind has been performed for the macroscopic binding energy of the nuclei, in the droplet model (Myers and Swiatecki 1970, 1974). This model takes account of the difference between the two density profiles of the neutrons and protons. Then, the total energy (volume energy, surface energy and Coulomb energy) for the actual nucleus with a diffuse surface is obtained from six shape-dependent functions: B_s, B_k, B_c, B_r, B_v and B_w, which are the surface function, the surface curvature function, the Coulomb and Coulomb redistribution functions and surface Coulomb functions (see table 1); these B functions being evaluated for an object with an equivalent sharp surface.

For an ellipsoid two of these functions are well known: the surface—older than one hundred years—and the Coulomb energy (see Kellog 1929). Carlson (1961) has introduced the effects of the anisotropy in the distribution of charges in this last energy.

Table 1. Shape dependences for arbitrary shapes (Myers and Swiatecki 1974). The B factors are dimensionless quantities that reduce to unity for spherical shapes. We take a unit radius for the sphere of equal volume: k_l is the mean local curvature of the surface, defined in terms of the principal radii of curvature R_1 and R_2, $k_l = R_1^{-1} + R_2^{-1}$.

$B_s = \int_\sigma \frac{d\sigma}{4\pi}$	$B_k = \int_\sigma \frac{k_l\, d\sigma}{8\pi}$	
$W(r) = \int_\tau \frac{d\tau'}{\lvert r - r'\rvert}$	$\bar{W} = \int_\tau \frac{W(r)\, d\tau}{(4\pi/3)}$	$\tilde{W}(r) = W(r) - \bar{W}$
$B_c = \frac{\bar{W}}{(8\pi/5)}$	$B_r = \int_\tau \frac{\lvert\tilde{W}(r)\rvert^2\, d\tau}{(64\pi^3/1575)}$	
$B_v = \int_\sigma \frac{\tilde{W}(r)\, d\sigma}{(-16\pi^2/15)}$	$B_w = \int_\sigma \frac{\lvert\tilde{W}(r)\rvert^2\, d\sigma}{(64\pi^3/225)}$	

Most of the shape-dependent functions usually found in the literature appear as limited developments of the deformation parameters. A comprehensive review of these formulae, not restricted to ellipsoidal shapes, will be found in Hasse (1971).

In our present work, as usual in electromagnetism or in probability theory, these functions are expressed in terms of the moments of the distribution. Our purpose is to show that the full use of the ellipsoidal symmetries and the reciprocal ellipsoid leads to very simple algebraic expressions for the shape-dependent factors. Then we will show that the total energy of a leptodermous ellipsoidal system is completely fixed by only one of these moments. We emphasise that most of our results are already known for the spheroidal shapes; our contribution aims at unifying the derivation of these results and extending them to asymmetrical shapes.

The ellipsoidal shape parametrisations are presented in § 2. The uses of the symmetries are described in § 3, which is devoted to the linear transformations of the elliptic integrals, and in § 4 where the reciprocal ellipsoid is defined. The deformations from the spherical shape have two main origins. The first is the finite size of the system that gives the surface and curvature energies. They are investigated in § 5. The second part of the deformation has a Coulomb origin and may include corrections due to the inhomogeneities of the charge distribution. This is the subject of § 7, the angular, surface and volume moments of the system being previously defined and calculated in § 6, using the symmetry properties. Finally, the evolution of the various shape-dependent functions is displayed for triaxial ellipsoidal shapes (one eccentricity being fixed).

2. Parametrisations of the ellipsoid

When the principal axes of an ellipsoid are coincident with the cartesian coordinate axes, the equation of its surface is given by

$$\sum_{i=1}^{3} (x_i/a_i)^2 = 1. \tag{1}$$

The a_i coefficients are the semi-axis lengths, and without loss of generality we can assume

$$a_1 \geqslant a_2 \geqslant a_3. \tag{2}$$

The volume enclosed by the ellipsoidal surface is

$$V = (4\pi/3)a_1a_2a_3.$$

If we start from a sphere with unit radius, the volume conservation implies

$$a_1a_2a_3 = 1. \tag{3}$$

The well fitted parametrisation for ellipsoidal shapes consists of the two squared eccentricities ε_1^2 and ε_3^2 (as defined below) that allow a complete description of all the shapes in a finite triangular cartesian plot $0 \leqslant \varepsilon_1^2 < 1$ and $\varepsilon_3^2 \leqslant \varepsilon_1^2$. These three eccentricities are

$$\varepsilon_1^2 = 1 - (a_3/a_1)^2, \tag{4}$$

$$\varepsilon_2^2 = 1 - (a_2/a_1)^2, \tag{5}$$

$$\varepsilon_3^2 = 1 - (a_3/a_2)^2. \tag{6}$$

The conditions (2) and (3) for the semi-axis lengths lay down the equivalent ones for the eccentricities:

$$\varepsilon_1 \geqslant \varepsilon_2, \qquad \varepsilon_1 \geqslant \varepsilon_3, \qquad \text{with } (1-\varepsilon_1^2) = (1-\varepsilon_2^2)(1-\varepsilon_3^2)$$

where the eccentricities are real positive numbers, smaller than unity.

We shall write

$$\sin\psi = \varepsilon_1, \qquad k = \varepsilon_2/\varepsilon_1, \qquad k' = \varepsilon_3/\varepsilon_1. \tag{7}$$

As is well known, most of the ellipsoidal geometrical properties can be expressed in terms of the incomplete elliptic integrals:

$$F(\psi, k) = \int_0^\psi \Delta^{-1}(\alpha, k)\, d\alpha, \tag{8}$$

$$E(\psi, k) = \int_0^\psi \Delta(\alpha, k)\, d\alpha, \tag{9}$$

where

$$\Delta(\alpha, k) = (1 - k^2 \sin^2\alpha)^{1/2}. \tag{10}$$

When the ellipsoid has an axis of revolution, the different parameters take the following values. In the prolate case

$$a_1 \geqslant a_2 = a_3, \qquad \varepsilon_3 = 0,\ \varepsilon_1 = \varepsilon_2, \qquad k = 1,\ k' = 0.$$

In the oblate case

$$a_1 = a_2 \geqslant a_3, \qquad \varepsilon_2 = 0,\ \varepsilon_1 = \varepsilon_3, \qquad k = 0,\ k' = 1.$$

In both cases, the incomplete elliptic integrals reduce to elementary functions. Another parametrisation will be very useful to determine the area and the mean local curvature of the ellipsoid:

$$x_1 = a_1 \sin v \cos u, \qquad x_2 = a_2 \sin v \sin u, \qquad x_3 = a_3 \cos v, \tag{11}$$

with $0 \leqslant v \leqslant \pi$ and $0 \leqslant u < 2\pi$.

3. Ellipsoidal symmetries and linear transformations

The algebraic description of the physical properties induces scalar expressions $\langle f(x_1, x_2, x_3)\rangle$ where the brackets stand for surface, angular or volume integrations (see for example, Chandrasekhar (1969), Rosenkilde (1967)). Each ellipsoid is defined unambiguously by the set of the semi-axis lengths, so that these expressions are uniquely functions of the three parameters a_1, a_2, a_3:

$$\langle f(x_1, x_2, x_3)\rangle = \mathscr{F}(a_1, a_2, a_3).$$

The a_i dependency of these $\mathscr{F}$ functions is often written in terms of the incomplete elliptic integrals: $E(\psi, k), F(\psi, k)$, $E(\psi, k')$ and $F(\psi, k')$, the arguments of which, (ψ, k, k'), are bound to the eccentricities as defined in equation (7).

Due to the symmetries of the ellipsoid, the physical properties of an ellipsoidal distribution of matter are invariant under a relabelling of the coordinate axes. When we know an analytic formula for a given expression $\langle f(x_1, x_2, x_3)\rangle$, this invariance enables us to write down all of a family of new expressions by interchanging the axis indices. The only difficulty is to know the effect of the axis relabelling on the incomplete elliptic integrals in the $\mathscr{F}$ expressions. Fortunately, these geometrical symmetries are connected with the linear transformations of the incomplete elliptic integrals.

In table 2 the linear transformations (Erdelyi *et al* 1953) are recalled. They give a set of elliptic arguments (ψ, k) so that any elliptic integrals having two of these arguments are connected by rational relations. Table 3 gives a geometric interpretation, showing the linear transformation that is induced by each relabelling of the axes for the two pairs of elliptic integrals $E(\psi, k)$, $F(\psi, k)$ and $E(\psi, k')$, $F(\psi, k')$.

4. The reciprocal ellipsoid

The equilibrium properties of ellipsoidal systems are very sensitive to the energy difference between prolate and oblate shapes at constant mean deformation. Thus we

Table 2. Linear transformations. $F \equiv F(\psi, k)$, $E \equiv E(\psi, k)$ and $\Delta \equiv \Delta(\psi, k)$ are defined by equations (9), (10) and (11). The dotted variables are the transformed ones.

Transformation number	$\sin\dot{\psi}$	$\dot{k}$	$F(\dot{\psi}, \dot{k})$
1	$k \sin\psi$	k^{-1}	kF
2	$-\mathrm{i}\tan\psi$	$(1-k^2)^{1/2}$	$-\mathrm{i}F$
3	$-\mathrm{i}(1-k^2)^{1/2}\tan\psi$	$(1-k^2)^{-1/2}$	$-\mathrm{i}(1-k^2)^{1/2}F$
4	$(1-k^2)^{1/2}\Delta^{-1}\sin\psi$	$\mathrm{i}k(1-k^2)^{-1/2}$	$(1-k^2)^{1/2}F$
5	$-\mathrm{i}k\Delta^{-1}\sin\psi$	$-\mathrm{i}k^{-1}(1-k^2)^{1/2}$	$-\mathrm{i}kF$

Transformation number	$E(\dot{\psi}, \dot{k})$
1	$k^{-1}[E-(1-k^2)F]$
2	$\mathrm{i}(E-F-\tan\psi\,\Delta)$
3	$\mathrm{i}(1-k^2)^{-1/2}[E-(1-k^2)F-\Delta\tan\psi)$
4	$(1-k^2)^{-1/2}(E-k^2\Delta^{-1}\sin\psi\cos\psi)$
5	$\mathrm{i}k^{-1}(E-F-k^2\Delta^{-1}\sin\psi\cos\psi)$

Table 3. Ellipsoid symmetries and induced linear transformations.

New axes			Transformation number on	
$\dot{a}_1$	$\dot{a}_2$	$\dot{a}_3$	$F(\psi, k)$, $E(\psi, k)$	$F(\psi, k')$, $E(\psi, k')$
a_2	a_1	a_3	4	1
a_1	a_3	a_2	1	4
a_3	a_2	a_1	2	2
a_2	a_3	a_1	5	3
a_3	a_1	a_2	3	5

define a transformation (cf figure 1) that conserves both the volume, and the coordinate lines (in the (u, v) parametrisation, see equation (11)), and transforms an oblate ellipsoid into a prolate one. If the point $M(x_1, x_2, x_3)$ describes the ellipsoid surface (1), its transform $M'(x'_1, x'_2, x'_3)$, defined by

$$x'_i = x_i/a_i^2,$$

generates the surface of the reciprocal ellipsoid, the equation of which is

$$\sum_i x_i'^2/(a_i^{-1})^2 = 1.$$

The distance p of the centre of the ellipsoid from the tangent plane, at the point M, is

$$p = \left(\sum_i (x_i^2/a_i^4)\right)^{-1/2} \tag{12}$$

while the unit outward normal $\boldsymbol{n}$ has components

$$n_i = x_i p/a_i^2 \tag{13}$$

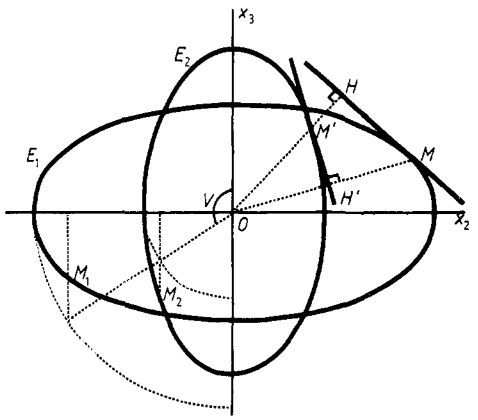

Figure 1. Illustration of the inversion leading to the reciprocal ellipsoid. The oblate ellipsoid E_2 of semi-axis lengths a_1^{-1}, a_2^{-1}, a_3^{-1} is the reciprocal of the prolate one E_1. The two ellipsoids have the same axis of revolution (x_2 axis on the figure). M' and H' are respectively the inverses of H and M. We notice that two reciprocal points M_1 and M_2 have the same coordinates (u, v) (see equation (11)); here $u = \frac{3}{2}\pi$.

so that

$$\boldsymbol{OM}' = \boldsymbol{n}/p.$$

This transformation is clearly involutive.

With the condition (3) the quantity

$$C = a_k^2 \, \mathrm{d}x_i \, \mathrm{d}x_j / x_k \qquad (i \neq j \neq k)$$

becomes the essential invariant of this transformation. At the point M the elementary solid angle and the surface element are given by

$$\mathrm{d}\omega = C/r^3 \qquad \text{and} \qquad \mathrm{d}\sigma = C/p \tag{14}$$

while at M'

$$\mathrm{d}\omega' = p^3 C \qquad \text{and} \qquad \mathrm{d}\sigma' = rC. \tag{15}$$

In reference to the parametrisation defined by (4), (5), and (6) and the linear transformations of the elliptic integrals, we must order the semi-axes:

$$a_1^* = a_3^{-1}, \qquad a_2^* = a_2^{-1}, \qquad a_3^* = a_1^{-1}, \tag{16}$$

so that

$$a_1^* \geqslant a_2^* \geqslant a_3^* \tag{17}$$

as in equation (2). Henceforth we shall use an asterisk to distinguish the expressions for the reciprocal ellipsoid (with the convention (17)) from those of the direct one. Naturally, one has still

$$r^* = 1/p \qquad \text{and} \qquad p^* = 1/r.$$

The transformations (16) affect the eccentricities:

$$\varepsilon_1^* = \varepsilon_1, \qquad \varepsilon_2^* = \varepsilon_3, \qquad \varepsilon_3^* = \varepsilon_2. \tag{18}$$

Furthermore

$$k'^* = k \qquad \text{and} \qquad k^* = k'. \tag{19}$$

Each expression for a given ellipsoid generates another one for the reciprocal one through this transformation, and

$$F^*(\psi, k) = F(\psi, k') \qquad \text{and} \qquad F^*(\psi, k') = F(\psi, k) \tag{20}$$

and the same for $E(\psi, k)$.

The reciprocal transformation will allow us to determine the surface moments of the ellipsoid (see § 6) as a function of the angular moments of the reciprocal ellipsoid. We shall see too, that the Coulomb properties of an ellipsoid are connected to the surface ones of the reciprocal ellipsoid (see § 7.2).

The most beautiful example is the connection between surface and mean curvature of the ellipsoid (see next section).

5. Surface and curvature functions

For a given i, all the points on the surface keeping n_i constant satisfy the equations (1) and (13). Their projection on the (x_j, x_k) coordinate plane is an ellipse:

$$(x_j/\alpha_j)^2+(x_k/\alpha_k)^2=1$$

with

$$\alpha_j^2(n_i)=a_j^2(1-n_i^2)[1-(1-a_i^2/a_j^2)n_i^2].$$

The α_j are the semi-axis lengths of the ellipse, the area of which is

$$S(n_i)=\pi\alpha_j(n_i)\alpha_k(n_i).$$

We define three quantities A_i ($i=1$, 2, and 3)

$$A_i=(4\pi)^{-1}\iint n_i^2\,\mathrm{d}\sigma=\langle n_i^2\rangle_\sigma. \tag{21}$$

Then the area A of the ellipsoid is given by

$$A=4\pi(A_1+A_2+A_3). \tag{22}$$

Due to the symmetries of the ellipsoid (cf § 3), we can restrict our attention to the particular case $i=3$, $j=1$ and $k=2$, and

$$A_3=\left(-\frac{1}{2\pi}\right)\int_{n_3=0}^{n_3=1} n_3\,\mathrm{d}S(n_3), \qquad A_3=\tfrac{1}{2}\int_0^1 \alpha_1(n_3)\alpha_2(n_3)\,\mathrm{d}n_3.$$

The reduction of the A_3 expression in terms of incomplete elliptic integrals is straightforward. With the help of the linear transformations, we obtain the three basic expressions

$$A_1=a_3[2a_1^2\varepsilon_1^3(1-k'^2)]^{-1}[(1-\varepsilon_1^2)E(\psi,k') +(\varepsilon_1^2-\varepsilon_3^2)F(\psi,k')-\varepsilon_1(1-\varepsilon_1^2)^{1/2}(1-\varepsilon_3^2)^{1/2}], \tag{23a}$$

$$A_2=a_3[2a_2^2\varepsilon_1^3k'^2(1-k'^2)]^{-1}[(1-k'^2)F(\psi,k') -(1-\varepsilon_3^2)E(\psi,k')+(1-\varepsilon_3^2)^{1/2}\varepsilon_1(1-\varepsilon_1^2)^{1/2}k'^2], \tag{23b}$$

$$A_3=(2a_3\varepsilon_1^3k'^2)^{-1}[E(\psi,k')-(1-\varepsilon_3^2)F(\psi,k')], \tag{23c}$$

where the condition (3) has been used.

The analytical expression of the ellipsoidal area is deduced from equations (22) and (23a, b, c),

$$A=2\pi a_1a_2[(\varepsilon_1^{-1}-\varepsilon_1)F(\psi,k')+\varepsilon_1E(\psi,k')+(1-\varepsilon_1^2)^{1/2}(1-\varepsilon_3^2)^{1/2}],$$

and the surface function B_s (see table 1) is

$$B_s=A/4\pi.$$

This result is obviously not new, but its derivation has allowed us to introduce the A_i (equation (21)) that are extensively used in the following. In cartesian coordinates the local mean curvature k_l is given by

$$k_l=\sum_i(a_i^2-x_i^2)p^3 \tag{24}$$

while the local Gauss curvature $1/R_1R_2$ is simply p^4 where p is defined by equation (12).

The direct integration of the local mean curvature k_1

$$K=\iint k_1\,\mathrm{d}\sigma$$

is not simple using equation (24). The result is more easily obtained by the (u, v) parametrisation:

$$\mathrm{d}\sigma = H\,\mathrm{d}u\,\mathrm{d}v \tag{25}$$

with

$$H^2=a_1^4\sin^2 v[(1-\varepsilon_2^2)-\varepsilon_1^2(1-\varepsilon_2^2)\sin^2 v+\varepsilon_2^2(1-\varepsilon_1^2)\sin^2 v\sin^2 u]$$

and

$$k_1=(a_1\sin v/H)^3a_2a_3(2-\varepsilon_2^2-\varepsilon_1^2\sin^2 v+\varepsilon_2^2\sin^2 v\sin^2 u).$$

Integrating first over u, we are left with an expression that can be reduced in terms of the incomplete elliptic integrals:

$$K=4\pi(a_2a_3/a_1)\{1+[\varepsilon_1^2(1-\varepsilon_1^2)(1-\varepsilon_2^2)]^{-1/2}[(1-\varepsilon_1^2)F(\psi, k)+\varepsilon_1^2E(\psi, k)]\}.$$

The curvature function B_k (see table 1) is

$$B_\mathrm{k}=K/8\pi.$$

With the help of the identities (18), (19) and (20) one can immediately see that

$$K=2A^*$$

and so

$$B_\mathrm{k}=B_\mathrm{s}^*.$$

One can notice that this important result is not a local property but just a mean one (see equations (14), (15) and (24)).

6. The ellipsoidal moments

When we study ellipsoidal distributions of matter, we are primarily interested in the calculation of the deviations from the pure, hard-edged, spherical distributions. Such properties can be derived in terms of the moments. For example small nuclear ellipsoidal deformations are quadrupole deformations. With the aim of studying Coulomb (or gravitational) effects either spread in the whole volume, or localised at the surface, we are led to define three kind of moments.

(i) Angular moments, such as

$$\langle x_i^n\rangle_\omega=(4\pi)^{-1}\iint (x_i)^n\,\mathrm{d}\omega$$

where ω stands for the solid angle.

(ii) Surface moments:

$$\langle x_i^n\rangle_\sigma=(4\pi)^{-1}\iint (x_i)^n\,\mathrm{d}\sigma$$

where $d\sigma$ is calculated in § 5, equation (25). The surface and angular moments are identical in the limiting case of the unit sphere.

(iii) Volume moments:

$$\langle x_i^n \rangle_\tau = (3/4\pi) \iiint (x_i)^n \, d\tau$$

where $d\tau$ is the elementary volume element, and the integration is carried out over the whole volume. The coefficients of the three integrals are chosen to give unity for the equivalent unit sphere and for $n = 0$.

6.1. *The angular moments*

In this study, we need only moments of order two ($n = 2$); they are bound by the simple relation

$$\sum_{i=1}^{3} \langle x_i^2 \rangle_\omega / a_i^2 = 1$$

and can be derived from $\langle r^2 \rangle_\omega$ that we define by

$$\langle r^2 \rangle_\omega = \sum_{i=1}^{3} \langle x_i^2 \rangle_\omega .$$

We shall use the method of Chandrasekhar (1969), which yields

$$\langle r^2 \rangle_\omega = a_2 a_3 \varepsilon_1^{-1} F(\psi, k). \tag{26}$$

The moments of order two are obtained using

$$\langle x_i^2 \rangle_\omega = \langle r^2 \rangle_\omega - a_i \, \partial \langle r^2 \rangle_\omega / \partial a_i . \tag{27}$$

By the use of equation (27), and with the derivatives of the incomplete elliptic integrals, the three moments of order two are written down. With the help of the linear transformations only one moment would be necessary to obtain these results.

$$\langle x_1^2 \rangle_\omega = a_2 a_3 / (\varepsilon_1^3 k^2)[F(\psi, k) - E(\psi, k)],$$

$$\langle x_2^2 \rangle_\omega = a_2 a_3 (1 - k^2 \varepsilon_1^2)/[\varepsilon_1^3 k^2 (1 - k^2)][E(\psi, k) - (1 - k^2) F(\psi, k) - a_3 \varepsilon_1 k^2 / a_2],$$

$$\langle x_3^2 \rangle_\omega = a_2 a_3 (1 - \varepsilon_1^2)/[\varepsilon_1^3 (1 - k^2)][a_2 \varepsilon_1 / a_3 - E(\psi, k)].$$

The angular moments are connected with the potential energy tensors for the ellipsoids, which Chandrasekhar (1969) has worked out, and if needed, moments of higher order can easily be generated. The basic $\langle x_i^2 \rangle_\omega$ angular moments may be developed into powers of k for deformations with small triaxiality. We readily find to the order two for the prolate shapes ($a_1 \geqslant a_2 \simeq a_3$, $k \to 1$)

$$\langle r^2 \rangle_\omega = a_3^2 (2\varepsilon_1)^{-1} \left[\ln \frac{1+\varepsilon_1}{1-\varepsilon_1} + \left(\frac{1+\varepsilon_1^2}{1-\varepsilon_1^2} \ln \frac{1+\varepsilon_1}{1-\varepsilon_1} - \frac{2\varepsilon_1}{1-\varepsilon_1^2} \right) \frac{(1-k^2)}{4} \right], \tag{28a}$$

$$\langle x_1^2 \rangle_\omega = a_3^2 (2\varepsilon_1^3)^{-1} \left[\ln \frac{1+\varepsilon_1}{1-\varepsilon_1} - 2\varepsilon_1 + \left(\frac{3-\varepsilon_1^2}{1-\varepsilon_1^2} \ln \frac{1+\varepsilon_1}{1-\varepsilon_1} - \frac{6\varepsilon_1}{1-\varepsilon_1^2} \right) \frac{(1-k^2)}{4} \right], \tag{28b}$$

$$\langle x_2^2 \rangle_\omega = a_3^2 (4\varepsilon_1^3)^{-1} \left[2\varepsilon_1 - (1-\varepsilon_1^2) \ln \frac{1+\varepsilon_1}{1-\varepsilon_1} - 3 \left(\frac{3+\varepsilon_1^2}{2} \ln \frac{1+\varepsilon_1}{1-\varepsilon_1} + \frac{\varepsilon_1^3 - 3\varepsilon_1}{1-\varepsilon_1^2} \right) \frac{(1-k^2)}{4} \right] \tag{28c}$$

$$\langle x_3^2\rangle_\omega = a_3^2(4\varepsilon_1^3)^{-1}\left[2\varepsilon_1-(1-\varepsilon_1^2)\ln\frac{1+\varepsilon_1}{1-\varepsilon_1}-\left(\frac{3+\varepsilon_1^2}{2}\ln\frac{1+\varepsilon_1}{1-\varepsilon_1}+\frac{\varepsilon_1^3-3\varepsilon_1}{1-\varepsilon_1^2}\right)\frac{(1-k^2)}{4}\right] \tag{28d}$$

and for the oblate shapes ($a_1 \simeq a_2 \geqslant a_3$, $k \to 0$)

$$\langle r^2\rangle_\omega = a_1a_3\varepsilon_1^{-1}\{\sin^{-1}\varepsilon_1+[(1-2\varepsilon_1^2)\sin^{-1}\varepsilon_1-\varepsilon_1(1-\varepsilon_1^2)^{1/2}]k^2/4\}, \tag{29a}$$

$$\langle x_1^2\rangle_\omega = a_1a_3(2\varepsilon_1^3)^{-1}\{\sin^{-1}\varepsilon_1-\varepsilon_1(1-\varepsilon_1^2)^{1/2} + [(3-4\varepsilon_1^2)\sin^{-1}\varepsilon_1-(3\varepsilon_1-2\varepsilon_1^3)(1-\varepsilon_1^2)^{1/2}]k^2/8\}, \tag{29b}$$

$$\langle x_2^2\rangle_\omega = a_1a_3(2\varepsilon_1^3)^{-1}\{\sin^{-1}\varepsilon_1-\varepsilon_1(1-\varepsilon_1^2)^{1/2} + 3[(3-4\varepsilon_1^2)\sin^{-1}\varepsilon_1-(3\varepsilon_1-2\varepsilon_1^3)(1-\varepsilon_1^2)^{1/2}]k^2/8\}, \tag{29c}$$

$$\langle x_3^2\rangle_\omega = a_3^3(a_1\varepsilon_1^3)^{-1}\{\varepsilon_1(1-\varepsilon_1^2)^{-1/2}-\sin^{-1}\varepsilon_1 + 3[(2\varepsilon_1^2/3-1)\sin^{-1}\varepsilon_1+\varepsilon_1(1-\varepsilon_1^2)^{1/2}]k^2/4\}. \tag{29d}$$

6.2. *The surface moments*

The evaluation of the surface moments implies the introduction of higher (than two) order expressions. Fixing our attention again on the following case $i=3$, $j=1$ and $k=2$, and introducing cross terms with an aim of completeness, we compute

$$\langle x_1^p x_2^q n_3^2\rangle_\sigma = (4\pi)^{-1}\iint x_1^p x_2^q n_3^2 \,\mathrm{d}\sigma.$$

We can follow the procedure of the area calculation:

$$\langle x_1^p x_2^q n_3^2\rangle_\sigma = -\tfrac{1}{2}\int_{n_3=0}^{n_3=1} n_3 \,\mathrm{d}\langle x_1^p x_2^q\rangle_{\sigma_0}$$

where $\langle x_1^p x_2^q\rangle_{\sigma_0}$ is the corresponding moment of the ellipse in the (x_1, x_2) plane; that is

$$\langle x_1^p x_2^q\rangle_{\sigma_0} = I_{pq}\alpha_1^{p+1}(n_3)\alpha_2^{q+1}(n_3)$$

with

$$I_{pq} = 4(p+q+2)^{-1}\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,\mathrm{d}\theta$$

and

$$\langle x_1^p n_2^q n_3^2\rangle_\sigma = (2\pi)^{-1}I_{pq}\int_0^1 \alpha_1^{p+1}(n_3)\alpha_2^{q+1}(n_3)\,\mathrm{d}n_3. \tag{30}$$

With the use of the ellipsoidal symmetries, we obtain the second-order expressions

$$\langle x_i^2 n_j^2\rangle_\sigma = \tfrac{1}{4}a_i^4(A_j-A_i)/(a_i^2-a_j^2), \qquad i\neq j. \tag{31}$$

Using the ellipsoidal surface equation, we deduce the diagonal term:

$$\langle x_i^2 n_i^2\rangle_\sigma = a_i^2(A_i-\langle x_j^2 n_i^2/a_j^2\rangle_\sigma-\langle x_k^2 n_i^2/a_k^2\rangle_\sigma), \qquad i\neq j\neq k. \tag{32}$$

Combining equations (31) and (32), the surface moments of second order follow:

$$\langle x_i^2\rangle_\sigma = (a_i^2/4)(2A_i+A_j+A_k), \qquad i\neq j\neq k,$$

$$\langle x_i^2\rangle_\sigma = (a_i^2/16\pi)(A+4\pi A_i).$$

The A_i may be expressed in terms of the angular moments $\langle x_i^2\rangle_\omega$, by the use of an important local equality, deduced from equations (14) and (15):

$$r^{*2}\,\mathrm{d}\omega' = p^2\,\mathrm{d}\sigma.$$

As

$$p^2 = \sum_i a_i^2 n_i^2$$

we have

$$\langle r^{*2}\rangle_\omega = \langle p^2\rangle_\sigma = \sum_i a_i^2 A_i = I. \tag{33}$$

I is naturally invariant under axis relabelling. $\langle r^2\rangle_\omega$ is a homogeneous function of degree one in a_i^2 (see equation (26)); then I is homogeneous of degree -1 in a_i^2.

Euler's theorem gives

$$\sum_i a_i^2 \frac{\partial I}{\partial a_i^2} = -I.$$

This relation holds whatever the a_i values are and, comparing with equation (33), we can write

$$A_i = -\partial I/\partial a_i^2. \tag{34}$$

Combining equations (33) and (34),we obtain

$$\langle x_i^{2*}\rangle_\omega = I - 2a_i^2 A_i.$$

This relation gives in a very simple way the analytical expressions of the A_i:

$$A_i = (2a_i^2)^{-1}\langle (r^2 - x_i^2)^*\rangle_\omega. \tag{35}$$

Thus, for axially symmetric shapes the reciprocal quantities $\langle (r^2 - x_i^2)^*\rangle_\omega$ are given by equations (28) and (29), noticing that the A_i functions for oblate shapes are deduced from the prolate angular moments and vice versa. Finally, the surface moments can be derived from the angular moments of the reciprocal ellipsoid by the relation

$$\langle x_i^2\rangle_\sigma = a_i^2 A/16\pi + \langle (r^2 - x_i^2)^*\rangle_\omega/8.$$

The algebraic simplicity of these equations is linkable to their order, which is less than or equal to the surface equation order. Moments of higher order can be deduced with a general recurrence formula derived from equation (30). The surface moments are closely related to the surface-energy tensors, which Rosenkilde (1967) has worked out in his studies of a rotating charged liquid drop.

6.3. The volume moments

The volume moments are considerably simpler to handle; by a homothetical transformation we can always reduce them to integrals over a spherical volume:

$$\langle x_i^n\rangle_\tau = 0, \qquad n \text{ odd},$$

$$\langle x_i^n\rangle_\tau = 3a_i^n[(n+1)(n+3)], \qquad n \text{ even}.$$

7. Shape-dependent Coulomb (or gravitational) functions

7.1. The Coulomb potential

Inside the ellipsoid, the Coulomb potential, induced by a homogeneous distribution of charges, is proportional to the function $W(\boldsymbol{r})$ (see table 1):

$$W(\boldsymbol{r}) = \int_\tau \frac{\mathrm{d}\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

As shown by Kellog (1929), $W(\boldsymbol{r})$ may be written as a quadratic function of the cartesian coordinates; using the notation of § 6,

$$W(\boldsymbol{r}) = 2\pi \sum_i \langle x_i^2\rangle_\omega \left(1 - \frac{x_i^2}{a_i^2}\right). \tag{36}$$

In droplet model formulae this function $W(\boldsymbol{r})$ is used to compute correction terms to the binding energy to take account either of the deviation of the proton distribution from its average value in the volume or of the redistribution of charges at the surface (Myers and Swiatecki 1974).

The mean value of $W(\boldsymbol{r})$ in the ellipsoid is written

$$\bar{W} = \left(\frac{3}{4\pi}\right) \int_\tau W(\boldsymbol{r})\, \mathrm{d}\tau.$$

With the results of § 6.3, $\bar{W}$ takes the simple form

$$\bar{W} = (8\pi/5)\langle r^2\rangle_\omega. \tag{37}$$

$\bar{W}$ is proportional to twice the Coulomb energy of the uniformly charged ellipsoid. The expression (37) is coherent with the theorem found by Carlson (1961), stating that the shape-dependent Coulomb energy of an ellipsoid $U(a_1, a_2, a_3)$ can be factorised:

$$U(a_1, a_2, a_3) = U(1, 1, 1)\langle r^2\rangle_\omega.$$

The Carlson theorem has a wider range of application because it applies to any distribution of charges inside the ellipsoid when the surfaces of constant density are a family of similar concentric ellipsoids.

Using equation (26), $\bar{W}$ takes the simple form

$$\bar{W} = (8\pi/5)a_2 a_3 \varepsilon_1^{-1} F(\psi, k).$$

The deviation of $W(\boldsymbol{r})$ from its average value is (see table 1 and equations (36) and (37))

$$\tilde{W}(\boldsymbol{r}) = 2\pi \sum_{i=1}^{3} \langle x_i^2\rangle_\omega (\tfrac{1}{5} - x_i^2/a_i^2)$$

and with the results of § 6, the following integral is deduced:

$$\int_\tau [\tilde{W}(\boldsymbol{r})]^2\, \mathrm{d}\tau = \left(\frac{32\pi^2}{525}\right)\left(5 \sum_i \langle x_i^2\rangle_\omega^2 - \langle r^2\rangle_\omega^2\right).$$

To study the Coulomb effects at the surface of the ellipsoidal distribution, we introduce the relative mean value of $W(\boldsymbol{r})$

$$\bar{W}_\mathrm{s} = \left(\frac{3}{16\pi^2}\right) \int_\sigma \mathrm{d}\sigma\, W_\mathrm{s}(\boldsymbol{r})$$

where $W_s(\boldsymbol{r})$ is $W(\boldsymbol{r})$ on the surface. This quantity is expressed in terms of the surface moments of § 6:

$$\bar{W}_s = (9/32\pi)A\langle r^2\rangle_\omega - \tfrac{3}{8}\sum_i A_i\langle x_i^2\rangle_\omega.$$

The redistribution function B_w (surface effects of second kind) is not calculated, since it involves surface moments of higher order and its contribution to the deformation energy has been found to be negligible (less than 1/10000 in realistic cases, cf Remaud (1978a)).

7.2. Shape-dependent functions

Collecting the results of the preceding sections, the shape-dependent Coulomb functions can be written analytically:

$$B_c = \langle r^2\rangle_\omega,$$

$$B_r = \tfrac{3}{2}\Big(5\sum_i \langle x_i^2\rangle_\omega^2 - \langle r^2\rangle_\omega^2\Big).$$

The last Coulomb function (B_v) is related to the former and to $\bar{W}_s$:

$$B_v = 6B_cB_s - 5\bar{W}_s;$$

then

$$B_v = \tfrac{3}{8}\Big(B_cB_s + 5\sum_i A_i\langle x_i^2\rangle_\omega\Big).$$

The surface function B_s (see § 5) is connected with the B_c Coulomb function by the relation

$$B_s = \tfrac{1}{2}B_c^*\Big(\sum_i a_i^{-2}\Big) - \tfrac{1}{2}\sum_i a_i^{-2}\langle x_i^{2*}\rangle_\omega$$

which derives from equation (35).

In figure 2, the evolution of the various B functions is displayed for triaxial shapes, the eccentricity ε_1 being fixed. B_s, B_k and B_r are sensitive to the asymmetry while B_c and B_v are very little dependent on a triaxial degree of freedom.

8. Conclusions

All the shape-dependent functions (except $\bar{W}_s$) are simple combinations of the $A_i(=\langle n_i^2\rangle_\sigma)$ and $\langle x_i^2\rangle_\omega$ functions. With the use of the reciprocal ellipsoid properties, only one set is necessary. With the help of the symmetry properties of elliptic integrals the last set has been deduced from one basic analytical expression. Then, as long as the deformations can be considered as ellipsoidal deformations and as long as the leptodermous approach is valid, the three analytical expressions $\langle x_i^2\rangle_\omega$ are sufficient to determine the macroscopic behaviour of a system such as a liquid drop or nucleus.

If we refer to nuclear physics, studies of collective spectra (Remaud 1978b, Kumar 1978) have shown that nuclear dynamics may depend strongly upon the macroscopic deformation energy. For example, at constant deformation magnitude an evolution in

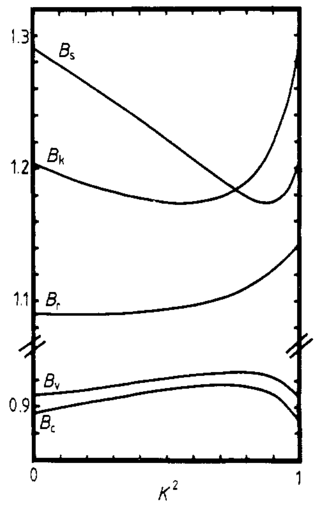

Figure 2. Evolution of the shape-dependent functions for asymmetric shapes. The value of the eccentricity ε_1 is 0.95. In the oblate limit case $k=0$, $a_1=a_2=1.474$, $a_3=0.460$; for the prolate shape $k=1$, $a_1=2.173$ and $a_2=a_3=0.678$.

the symmetry of a nucleus has strong effects on the spectrum structure, although this transformation is often almost energy degenerated. A large class of nuclei has been labelled as soft, since their wavefunction of shape collective coordinates extends itself over a large domain in the deformation space; the spectroscopic properties of these nuclei are then dependent on the precise evaluation of the potential energy for an extended range of deformations and mainly of the triaxial degree of freedom. In that sense, our study is helpful since it provides analytical results for all deformations.

Furthermore, the relationship that we have derived between the curvature function of an ellipsoid and the surface function of its reciprocal may provide clues for the study of the curvature effects on the macroscopical nucleus energy.

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